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Semigroups and Stochastic Processes associated with functions of the Lévy Laplacian

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Abstract

In this paper, we discuss equi-continuous semigroups of class (C_0) and infinite dimensional stochastic processes generated by functions of the Lévy Laplacian following our recent results.

1. Introduction

An infinite dimensional Laplacian, the Lévy Laplacian, was introduced by P. Lévy [17]. This Laplacian was discussed in the framework of white noise analysis initiated by T. Hida [4]. L. Accardi et al. [1] obtained an important relationship between this Laplacian and the Yang-Mills equations. It has been studied by many authors (see [1, 2, 3, 5, 7, 8, 13, 15, 16, 18, 21, 22, 23, 24 etc]).

In the previous papers [25,26] we obtained stochastic processes generated by the powers of an extended Lévy Laplacian and also in [29] we obtained stochastic processes generated by some functions of the Laplacian.

The purpose of this paper is to present recent developments on stochastic processes associated with functions of the Lévy Laplacian acting on white noise distributions based on the idea in [26,27,29,30].

The paper is organized as follows. In Section 2 we summarize some basic definitions and results in white noise analysis. In Section 3 we introduce a Hilbert space as a domain of the extended Lévy Laplacian which is self-adjoint on the domain following our previous paper [27], and we give an equi-continuous semigroup of class (C_0) generated by some functions of the extended Lévy Laplacian. In Section 4 we give infinite dimensional stochastic processes generated by those functions of the Lévy Laplacian. In the last section we give a homeomorphism to connect the Number operator to the Lévy Laplacian and also give a relationship between the semigroup generated by the Lévy Laplacian and an infinite dimensional Ornstein-Uhlenbeck process.

2. Preliminaries

In this section we assemble some basic notations of white noise analysis following [7, 12, 15, 19].

We take the space $E^* \equiv \mathcal{S}'(\mathbf{R})$ of tempered distributions with the standard Gaussian measure μ which satisfies

$$\int_{E^*} \exp\{i\langle x, \xi\rangle\}\, d\mu(x) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \qquad \xi \in E \equiv \mathcal{S}(\mathbf{R}),$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^* \times E$.

Let $A = -(d/du)^2 + u^2 + 1$. This is a densely defined self-adjoint operator on $L^2(\mathbf{R})$ and there exists an orthonormal basis $\{e_\nu; \nu \geq 0\}$ for $L^2(\mathbf{R})$ such that $Ae_\nu = 2(\nu+1)e_\nu$. We define the norm $|\cdot|_p$ by $|f|_p = |A^p f|_0$ for $f \in E$ and $p \in \mathbf{R}$, where $|\cdot|_0$ is the $L^2(\mathbf{R})$- norm, and let E_p be the completion of E with respect to the norm $|\cdot|_p$. Then E_p ia a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space E_p' of E_p is the same as E_{-p} (see [10]).

Let E be the projective limit space of $\{E_p; p \geq 0\}$ and E^* the dual space of E. Then E becomes a nuclear space with the Gel'fand triple $E \subset L^2(\mathbf{R}) \subset E^*$. We denote the complexifications of $L^2(\mathbf{R})$, E and E_p by $L^2_{\mathbf{C}}(\mathbf{R})$, $E_{\mathbf{C}}$ and $E_{\mathbf{C},p}$, respectively.

The space $(L^2) = L^2(E^*, \mu)$ of complex-valued square-integrable functionals defined on E^* admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbf{N}$ and $H_0 = \mathbf{C}$. Let $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$ denote the n-fold symmetric tensor product of $L^2_{\mathbf{C}}(\mathbf{R})$. If $\varphi \in (L^2)$ has the representation $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n)$, $f_n \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$, then the (L^2)-norm $\|\varphi\|_0$ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{1/2},$$

where $|\cdot|_0$ is the norm of $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$.

For $p \in \mathbf{R}$, let $\|\varphi\|_p = \|\Gamma(A)^p \varphi\|_0$, where $\Gamma(A)$ is the second quantization operator of A. If $p \geq 0$, let $(E)_p$ be the domain of $\Gamma(A)^p$. If $p < 0$, let $(E)_p$ be the completion of (L^2) with respect to the norm $\|\cdot\|_p$. Then $(E)_p$, $p \in \mathbf{R}$, is a Hilbert space with the norm $\|\cdot\|_p$. It is easy to see that for $p > 0$, the dual space $(E)_p^*$ of $(E)_p$ is given by $(E)_{-p}$. Moreover, for any $p \in \mathbf{R}$, we have the decomposition

$$(E)_p = \bigoplus_{n=0}^{\infty} H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in E_{\mathbf{C}}^{\hat{\otimes} n}\}$ with respect to $\|\cdot\|_p$. Here $E_{\mathbf{C}}^{\hat{\otimes} n}$ is the n-fold symmetric tensor product of $E_{\mathbf{C}}$. We also have $H_n^{(p)} = \{\mathbf{I}_n(f); f \in E_{\mathbf{C},p}^{\hat{\otimes} n}\}$ for any $p \in \mathbf{R}$, where

$E_{\mathbf{C},p}^{\hat{\otimes}n}$ is also the n-fold symmetric tensor product of $E_{\mathbf{C},p}$. The norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)_p$ is given by

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty} n! |f_n|_p^2 \right)^{1/2}, \quad f_n \in E_{\mathbf{C},p}^{\hat{\otimes}n},$$

where the norm of $E_{\mathbf{C},p}^{\hat{\otimes}n}$ is denoted also by $|\cdot|_p$.

The projective limit space (E) of spaces $(E)_p$, $p \in \mathbf{R}$ is a nuclear space. The inductive limit space $(E)^*$ of spaces $(E)_p, p \in \mathbf{R}$ is nothing but the dual space of (E). The space $(E)^*$ is called the space of *generalized white noise functionals.* We denote by $\ll \cdot, \cdot \gg$ the canonical bilinear form on $(E)^* \times (E)$. Then we have

$$\ll \Phi, \varphi \gg = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

for any $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (E)^*$ and $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)$, where the canonical bilinear form on $(E_{\mathbf{C}}^{\hat{\otimes}n})^* \times (E_{\mathbf{C}}^{\hat{\otimes}n})$ is denoted also by $\langle \cdot, \cdot \rangle$.

Since $\exp\langle \cdot, \xi \rangle \in (E)$, the *S-transform* is defined on $(E)^*$ by

$$S[\Phi](\xi) = \exp\left(-\frac{1}{2} \langle \xi, \xi \rangle \right) \ll \Phi, \exp\langle \cdot, \xi \rangle \gg, \qquad \xi \in E_{\mathbf{C}}.$$

3. An equi-continuous semigroup of class (C_0) generated by a function of the Lévy Laplacian

Let Φ be in $(E)^*$. Then the S-transform $S[\Phi]$ of Φ is *Fréchet differentiable,* i.e.

$$S[\Phi](\xi + \eta) = S[\Phi](\xi) + S[\Phi]'(\xi)(\eta) + o(\eta),$$

where $o(\eta)$ means that there exists $p \geq 0$ depending on ξ such that $o(\eta)/|\eta|_p \to 0$ as $|\eta|_p \to 0$.

We fix a finite interval T in $\mathbf{R}$. Take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty} \subset E$ for $L^2(T)$ satisfying the equally dense and uniform boundedness property (see [7,15,16,18,24, etc]). Let $\mathcal{D}_L$ denote the set of all $\Phi \in (E)^*$ such that the limit

$$\tilde{\Delta}_L S[\Phi](\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi]''(\xi)(\zeta_n, \zeta_n)$$

exists for any $\xi \in E_{\mathbf{C}}$ and is in $S[(E)^*]$. The Lévy Laplacian Δ_L is defined by

$$\Delta_L \Phi = S^{-1} \tilde{\Delta}_L S \Phi$$

for $\Phi \in \mathcal{D}_L$. We denote the set of all functionals $\Phi \in \mathcal{D}_L$ such that $S[\Phi](\eta) = 0$ for all $\eta \in E$ with $\text{supp}(\eta) \subset T^c$ by $\mathcal{D}_L^T$.

A generalized white noise functional

$$\Phi = \int_{\mathbf{R}^n} f(u_1, \dots, u_n) : e^{ia_1 x(u_1)} \cdots e^{ia_n x(u_n)} : d\mathbf{u} \in \mathcal{D}_L^T, \tag{3.1}$$

$$f \in L^1_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n} \cap L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}, a_k \in \mathbf{R}, k = 1, 2, \dots, n,$$

is equal to

$$\int_{T^n} f(u_1, \dots, u_n) : e^{ia_1 x(u_1)} \cdots e^{ia_n x(u_n)} : d\mathbf{u}$$

and the S-transform $S[\Phi]$ of Φ is given by

$$S[\Phi](\xi) = \int_{T^n} f(\mathbf{u}) e^{ia_1 \xi(u_1)} \cdots e^{ia_n \xi(u_n)} d\mathbf{u}. \tag{3.2}$$

This functional is important as an eigenfunction of the operator Δ_L. In fact, we have the following result:

Theorem 1.[27] *A generalized white noise functional Φ as in (3.1) satisfies the equation*

$$\Delta_L \Phi = -\frac{1}{|T|} \sum_{k=1}^{n} a_k^2 \Phi. \tag{3.3}$$

We set

$$\mathbf{D}_n = \left\{ \int_{T^n} f(\mathbf{u}) : \prod_{\nu=1}^{n} e^{ix(u_\nu)} : d\mathbf{u} \in \mathcal{D}_L^T; f \in E_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n} \right\}$$

for each $n \in \mathbf{N}$ and set $\mathbf{D}_0 = \mathbf{C}$. Then $\mathbf{D}_n$ is a linear subspace of $(E)_{-p}$ for any $p \geq 1$, and Δ_L is a linear operator from $\mathbf{D}_n$ into itself such that $\|\Delta_L \Phi\|_{-p} = \frac{n}{|T|} \|\Phi\|_{-p}$ for any $\Phi \in \mathbf{D}_n$. We define a space $\overline{\mathbf{D}_n}$ by the completion of $\mathbf{D}_n$ in $(E)_{-p}$ with respect to $\|\cdot\|_{-p}$. Then for each $n \in \mathbf{N} \cup \{0\}$, $\overline{\mathbf{D}_n}$ becomes a Hilbert space with the inner product of $(E)_{-p}$. For each $n \in \mathbf{N} \cup \{0\}$, the operator Δ_L can be extended to a continuous linear operator $\overline{\Delta_L}$ from $\overline{\mathbf{D}_n}$ into itself satisfying

$$\|\overline{\Delta_L} \Phi\|_{-p} = \frac{n}{|T|} \|\Phi\|_{-p} \text{ for any } \Phi \in \overline{\mathbf{D}_n}.$$

The operator $\overline{\Delta_L}$ is a self-adjoint operator on $\overline{\mathbf{D}_n}$ for each $n \in \mathbf{N} \cup \{0\}$.

Proposition 2. [27] *Let $\Phi = \sum_{n=0}^{\infty} \Phi_n, \Psi = \sum_{n=0}^{\infty} \Psi_n$ be generalized white noise functionals such that Φ_n and Ψ_n are in $\overline{\mathbf{D}_n}$ for each $n \in \mathbf{N} \cup \{0\}$. If $\Phi = \Psi$ in $(E)^*$, then $\Phi_n = \Psi_n$ in $(E)^*$ for each $n \in \mathbf{N} \cup \{0\}$.*

Let $\alpha_N(n) = \sum_{\ell=0}^{N} \left(\frac{n}{|T|}\right)^{2\ell}$. Proposition 2 says that $\sum_{n=0}^{\infty} \Phi_n, \Phi_n \in \overline{\mathbf{D}_n}$, is uniquely determined as an element of $(E)^*$. Therefore, for any $N \in \mathbf{N} \cup \{0\}$, we can define a space $\mathbf{E}_{-p,N}$ by

$$\mathbf{E}_{-p,N} = \left\{ \sum_{n=0}^{\infty} \Phi_n \in (E)^*; \sum_{n=0}^{\infty} \alpha_N(n) \|\Phi_n\|_{-p}^2 < \infty, \Phi_n \in \overline{\mathbf{D}_n}, n = 0, 1, 2, \dots \right\}$$

with the norm $|||\cdot|||_{-p,N}$ given by

$$|||\Phi|||_{-p,N} = \left(\sum_{n=0}^{\infty} \alpha_N(n) \|\Phi_n\|_{-p}^2\right)^{1/2}, \quad \Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,N}$$

for each $N \in \mathbf{N} \cup \{0\}$ and $p \geq 1$. For any $N \in \mathbf{N} \cup \{0\}$ and $p \geq 1$, $\mathbf{E}_{-p,N}$ is a Hilbert space with the norm $|||\cdot|||_{-p,N}$.

Put $\mathbf{E}_{-p,\infty} = \bigcap_{N\geq 1} \mathbf{E}_{-p,N}$ with the projective limit topology and define $\mathbf{E}_{-p,-\infty}$ by its dual space. If we introduce a Hilbert space $\mathbf{E}_{-p,-N}$ by the dual space of $\mathbf{E}_{-p,N}$ with the norm $|||\cdot|||_{-p,-N}$ given by

$$|||\Phi|||_{-p,-N} = \left(\sum_{n=0}^{\infty} \alpha_N(n)^{-1} \|\Phi_n\|_{-p}^2\right)^{1/2}, \quad \Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,-N}.$$

Then, for any $N \geq 1$, we have the following inclusion relations:

$$\mathbf{E}_{-p,\infty} \subset \mathbf{E}_{-p,N+1} \subset \mathbf{E}_{-p,N} \subset \mathbf{E}_{-p,1} \subset (E)_{-p} \subset \mathbf{E}_{-p,-1} \subset \mathbf{E}_{-p,-N} \subset \mathbf{E}_{-p,-N-1} \subset \mathbf{E}_{-p,-\infty}.$$

The space $\mathbf{E}_{-p,\infty}$ includes $\overline{\mathbf{D}}_n$ for any $n \in \mathbf{N} \cup \{0\}$. The operator $\overline{\Delta_L}$ can be extended to a continuous linear operator defined on $\mathbf{E}_{-p,-\infty}$, denoted by the same notation $\overline{\Delta_L}$, satisfying $|||\overline{\Delta_L}\Phi|||_{-p,N} \leq |||\Phi|||_{-p,N+1}$, $\Phi \in \mathbf{E}_{-p,N+1}$, for each $N \in \mathbf{Z}^* \equiv \mathbf{Z} \setminus (-1,1)$. Any restriction of $\overline{\Delta_L}$ is also denoted by the same notation $\overline{\Delta_L}$. With these properties, we have the following:

Theorem 3. *The operator $\overline{\Delta_L}$ restricted on $\mathbf{E}_{-p,N+1}$ is a self-adjoint operator densely defined on $\mathbf{E}_{-p,N}$ for each $N \in \mathbf{Z}^*$ and $p \geq 1$.*

Proof. We can apply the same proof of Theorem 2 in [27] to this theorem. □

Let $\{X_t; t \geq 0\}$ be a stochastic process and $c_{X_t}(z)$ be a characteristic function of X_t. For each $t \geq 0$ we consider an operator $G[X_t]$ on $\mathbf{E}_{-p,-\infty}$ defined by

$$G[X_t]\Phi = \sum_{n=0}^{\infty} c_{X_t}\left(\frac{n}{|T|}\right) \Phi_n$$

for $\Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,-\infty}$. For any $\Phi = \sum_{n=0}^{\infty} \Phi_n$ in $\mathbf{E}_{-p,-\infty}$, there exists $N \in \mathbf{Z}^*$ such that $\Phi \in \mathbf{E}_{-p,N}$. Then, for any $t \geq 0, p \geq 1$, the norm $|||G[X_t]\Phi|||_{-p,N}$ is estimated as follows:

$$\begin{aligned} |||G[X_t]\Phi|||_{-p,N}^2 &= \sum_{n=0}^{\infty} \alpha_N(n) \left\| c_{X_t}\left(\frac{n}{|T|}\right) \Phi_n \right\|_{-p}^2 \\ &\leq \sum_{n=0}^{\infty} \alpha_N(n) \|\Phi_n\|_{-p}^2 = |||\Phi|||_{-p,N}^2, \end{aligned}$$

where $\alpha_N(n)$ means $\alpha_{-N}(n)^{-1}$ for $N \leq 0$.

Thus the operator $G[X_t]$ is a continuous linear operator from $\mathbf{E}_{-p,-\infty}$ into itself. Moreover we have the following:

Proposition 4. *Let $\{X_t; t \geq 0\}$ be a stochastic process. Then the family $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) if and only if there exists a complex-valued continuous function $h(z)$ of $z \in \mathbf{R}$ such that $h(0) = 0$ and $c_{X_t}(z) = e^{h(z)t}$ for all $t \geq 0$.*

Proof. If there exists a complex-valued continuous function $h(z)$ of $z \in \mathbf{R}$ such that $c_{X_t}(z) = e^{h(z)t}$, then it is easily checked that $G[X_0] = I$, $G[X_t]G[X_s] = G[X_{t+s}]$ for each $t, s \geq 0$. Moreover we can estimate that

$$\begin{aligned} |||G[X_t]\Phi - G[X_{t_0}]\Phi|||_{-p,N}^2 &= \sum_{n=0}^{\infty} \alpha_N(n) \left| c_{X_t}\left(\frac{n}{|T|}\right) - c_{X_{t_0}}\left(\frac{n}{|T|}\right) \right|^2 ||\Phi_n||_{-p}^2 \\ &\leq 4 \sum_{n=0}^{\infty} \alpha_N(n) ||\Phi_n||_{-p}^2 = 4|||\Phi|||_{-p,N}^2 < \infty \end{aligned}$$

for each $t, t_0 \geq 0, N \in \mathbf{Z}^*$ and $\Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,N}$. Therefore, by the Lebesgue convergence theorem, we get that

$$\lim_{t \to t_0} G[X_t]\Phi = G[X_{t_0}]\Phi \quad \text{in} \quad \mathbf{E}_{-p,\infty}$$

for each $t_0 \geq 0$ and $\Phi \in \mathbf{E}_{-p,-\infty}$. Thus the family $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0). Conversely, if $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0), then it is easily checked that $c_{X_0}\left(\frac{n}{|T|}\right) = 1$, $c_{X_t}\left(\frac{n}{|T|}\right) c_{X_s}\left(\frac{n}{|T|}\right) = c_{X_{t+s}}\left(\frac{n}{|T|}\right)$ for any $t, s \geq 0$ and $\lim_{t \to t_0} c_{X_t}\left(\frac{n}{|T|}\right) = c_{X_{t_0}}\left(\frac{n}{|T|}\right)$ for any $t_0 \geq 0$ and $n \in \mathbf{N}$. Therefore, by the continuity of $c_{X_t}(z)$ of z, we have that $c_{X_0} = 1$, $c_{X_t} c_{X_s} = c_{X_{t+s}}$ for any $t, s \geq 0$ and $\lim_{t \to t_0} c_{X_t} = c_{X_{t_0}}$ for any $t_0 \geq 0$. Consequently, there exists a complex-valued function $h(z)$ of $z \in \mathbf{R}$ such that $h(0) = 0$ and $c_{X_t}(z) = e^{h(z)t}$. Since $c_{X_t}(z)$ is a characteristic function, the function $h(z)$ is continuous. □

For any $p \geq 1$ and complex-valued continuous function $h(z), z \in \mathbf{R}$ satisfying the condition:

(P) there exists a polynomial $r(z)$ of $z \in \mathbf{R}$ such that $|h(z)| \leq r(|z|)$ for all $z \in \mathbf{R}$,

the operator $h(-\overline{\Delta_L})$ on $\mathbf{E}_{-p,-\infty}$ is given by

$$h(-\overline{\Delta_L})\Phi = \textstyle\sum_{n=0}^{\infty} h\left(\frac{n}{|T|}\right) \Phi_n, \text{ for } \Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,-\infty}.$$

Theorem 5. *If $h(z)$ in Proposition 4 satisfies the condition* **(P)**, *then the infinitesimal generator of $\{G[X_t]; t \geq 0\}$ is given by $h(-\overline{\Delta_L})$.*

Proof. Let $p \geq 1$ and let $\Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}_{-p,-\infty}$. Then, there exists $N \in \mathbf{Z}^*$ such that $\Phi \in \mathbf{E}_{-p,N}$. Let d_r be the degree of the polynomial r in the condition **(P)**. Then we note that

$$\left|\left|\left| \frac{G[X_t]\Phi - \Phi}{t} - h(-\overline{\Delta_L})\Phi \right|\right|\right|_{-p,N-d_r}^2 = \sum_{n=0}^{\infty} \alpha_{N-d_r}(n) \left|\left| \left(\frac{e^{h\left(\frac{n}{|T|}\right)t} - 1}{t} - h\left(\frac{n}{|T|}\right) \right) \Phi_n \right|\right|_{-p}^2 \tag{3.4}$$

Since $e^{h(z)t}$ is a characteristic function, we note that $Re[h(z)] \leq 0$. By the mean value theorem, for any $t > 0$ there exists a constant $\theta \in (0,1)$ such that

$$\left|\frac{e^{h\left(\frac{n}{|T|}\right)t}-1}{t}\right| = \left|h\left(\frac{n}{|T|}\right)\right| e^{Re\left[h\left(\frac{n}{|T|}\right)\right]t\theta} \leq r\left(\frac{n}{|T|}\right).$$

Therefore we get that

$$\begin{aligned} \left\|\frac{e^{h\left(\frac{n}{|T|}\right)t}-1}{t}\Phi_n - h\left(\frac{n}{|T|}\right)\Phi_n\right\|_{-p}^2 &= \left|\frac{e^{h\left(\frac{n}{|T|}\right)t}-1}{t} - h\left(\frac{n}{|T|}\right)\right|^2 \|\Phi_n\|_{-p}^2 \\ &\leq 4r\left(\frac{n}{|T|}\right)^2 \|\Phi_n\|_{-p}^2. \end{aligned}$$

Since there exists a positive constant C_r depending on r such that $\alpha_{N-d_r}(n)r\left(\frac{n}{|T|}\right)^2 \leq C_r\alpha_N(n)$, we have

$$\sum_{n=0}^{\infty} \alpha_{N-d_r}(n) r\left(\frac{n}{|T|}\right)^2 \|\Phi_n\|_{-p}^2 < \infty. \tag{3.5}$$

By (3.4), (3.5) and

$$\lim_{t\to 0}\left|\frac{e^{h\left(\frac{n}{|T|}\right)t}-1}{t} - h\left(\frac{n}{|T|}\right)\right| = 0,$$

the Lebesgue convergence theorem admits

$$\lim_{t\to 0}\left|\left|\left|\frac{G[X_t]\Phi-\Phi}{t} - h(-\overline{\Delta_L})\Phi\right|\right|\right|_{-p,N-d_r}^2 = 0.$$

Thus the proof is completed. □

4. Stochastic processes generated by functions of the Lévy Laplacian

In this section, we give stochastic processes generated by functions of the extended Lévy Laplacian by considering the stochastic expression of the operator $G[X_t]$.

Let $\{X_t; t \geq 0\}$ be a stochastic process such that $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) and satisfies the condition of Theorem 5. Take a smooth function $\eta_T \in E$ with $\eta_T = \frac{1}{|T|}$ on T. Put $\widetilde{G[X_t]} = SG[X_t]S^{-1}$ on $S[\mathbf{E}_{-p,\infty}]$ with the topology induced from $\mathbf{E}_{-p,\infty}$ by the S-transform. Then by Theorem 5, $\{\widetilde{G[X_t]}; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by the operator $h(-\widetilde{\Delta_L})$, where $\widetilde{\Delta_L}$ means $S\overline{\Delta_L}S^{-1}$.

Let $\{\mathbf{X}_t; t \geq 0\}$ be an E-valued stochastic process given by $\mathbf{X}_t = \xi + X_t\eta_T$, $\xi \in E$. Then we have the following:

Theorem 6. *Let F be the S-transform of a generalized white noise functional in $\mathbf{E}_{-p,\infty}$. Then it holds that*

$$\widetilde{G[X_t]}F(\xi) = \mathrm{E}[F(\mathbf{X}_t)|\mathbf{X}_0 = \xi], \quad \xi \in E.$$

Proof. Put $F(\xi) = \int_{T^n} f(\mathbf{u})e^{i\xi(u_1)} \cdots e^{i\xi(u_n)}d\mathbf{u}$ with $f \in E_{\mathbf{C}}^{\hat{\otimes}n}$. Then we have

$$\begin{aligned}
\mathrm{E}[F(\mathbf{X}_t)|\mathbf{X}_0 = \xi] &= \mathrm{E}[F(\xi + X_t\eta_T)] \\
&= \int_{T^n} f(\mathbf{u})e^{i\xi(u_1)} \cdots e^{i\xi(u_n)}\mathrm{E}[e^{i\frac{n}{|T|}X_t}]d\mathbf{u} \\
&= e^{h\left(\frac{n}{|T|}\right)t}F(\xi) = \widetilde{G[X_t]}F(\xi).
\end{aligned}$$

Let $F = \sum_{n=0}^{\infty} F_n \in S[\mathbf{E}_{-p,\infty}]$. Then for any $n \in \mathbf{N}\cup\{0\}$, F_n is expressed in the following form:

$$F_n(\xi) = \lim_{N\to\infty} \int_{T^n} f^{[N]}(\mathbf{u})e^{i\xi(u_1)} \cdots e^{i\xi(u_n)}d\mathbf{u},$$

where $(f^{[N]})_N$ is a sequence of functions in $E_{\mathbf{C}}^{\hat{\otimes}n}$. Hence we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathrm{E}[|F_n(\xi + X_t\eta_T)|] \\
= &\sum_{n=0}^{\infty} \mathrm{E}\left[\lim_{N\to\infty} \left|\int_{T^n} f^{[N]}(\mathbf{u})e^{i\xi(u_1)} \cdots e^{i\xi(u_n)}e^{iX_t\eta_T(u_1)} \cdots e^{iX_t\eta_T(u_n)}d\mathbf{u}\right|\right] \\
= &\sum_{n=0}^{\infty} \lim_{N\to\infty} \left|\int_{T^n} f^{[N]}(\mathbf{u})e^{i\xi(u_1)} \cdots e^{i\xi(u_n)}d\mathbf{u}\right| \\
= &\sum_{n=0}^{\infty} |F_n(\xi)|.
\end{aligned}$$

Since $F_n \in S[\mathbf{E}_{-p,\infty}]$, there exists some $\Phi_n \in \mathbf{E}_{-p,\infty}$ such that $F_n = S[\Phi_n]$ for any n. By the characterization theorem of the U-functional (see [12,20,21, etc]), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} |F_n(\xi)| &\le \sum_{n=0}^{\infty} ||\Phi_n||_{-p}||\varphi_\xi||_p \\
&\le \left\{\sum_{n=0}^{\infty} \alpha_N(n)^{-1}\right\}^{1/2} \left\{\sum_{n=0}^{\infty} \alpha_N(n)||\Phi_n||_{-p}^2\right\}^{1/2} ||\varphi_\xi||_p < \infty,
\end{aligned}$$

for all $\xi \in E$ and some $N \ge 1$, where $\varphi_\xi(x) =: \exp\{\langle x, \xi\rangle\} :$. Therefore by the continuity of $\widetilde{G[X_t]}$ we get that

$$\begin{aligned}\mathrm{E}[F(\xi + X_t\eta_T)] &= \sum_{n=0}^{\infty}\mathrm{E}[F_n(\xi + X_t\eta_T)] \\ &= \sum_{n=0}^{\infty}\widetilde{G[X_t]}F_n(\xi) \\ &= \widetilde{G[X_t]}F(\xi).\end{aligned}$$

Thus we obtain the assertion. □

Theorem 6 says that the infinite dimensional stochastic process $\{\mathbf{X}_t; t \geq 0\}$ is generated by $h(-\widetilde{\Delta_L})$.

For any $\Phi \in (E)^*$ and $\eta \in E$, the translation $\tau_\eta\Phi$ of Φ by η is defined as a generalized white noise functional $\tau_\eta\Phi$ whose S-transform is given by $S[\tau_\eta\Phi](\xi) = S[\Phi](\xi + \eta)$, $\xi \in E_{\mathbf{C}}$. Then we can translate Theorem 6 to be in words of generalized white noise functionals.

Corollary 7. *Let Φ be a generalized white noise functional in $\mathbf{E}_{-p,\infty}$. Then it holds that*

$$G[X_t]\Phi(x) = \mathrm{E}[\tau_{X_t\eta_T}\Phi(x)].$$

By Corollary 7 we can see that $\{\tau_{X_t\eta_T}; t \geq 0\}$ is an operator-valued stochastic process and $\{E[\tau_{X_t\eta_T}]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by $h(-\overline{\Delta_L})$.

Example: Let $\{X_t; t \geq 0\}$ be an additive process with the characteristic function $c_{X_t}(z)$ of X_t for each $t \geq 0$ given by

$$c_{X_t}(z) = \exp\left[t\left\{imz - \frac{v}{2}z^2 + \int_{|u|<1}\left(e^{izu} - 1 - izu\right)d\nu(u) + \int_{|u|\geq 1}\left(e^{izu} - 1\right)d\nu(u)\right\}\right],$$

where $m \in \mathbf{R}, v \geq 0$ and ν is a measure on $\mathbf{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbf{R}}(1 \wedge |u|^2)d\nu(u) < \infty$. Then the function

$$h(z) = imz - \frac{v}{2}z^2 + \int_{|u|<1}\left(e^{izu} - 1 - izu\right)d\nu(u) + \int_{|u|\geq 1}\left(e^{izu} - 1\right)d\nu(u)$$

satisfies conditions of Proposition 5 and the condition $(\mathbf{P})$. Therefore $\{G[X_t]; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by $h(-\overline{\Delta_L})$. The stochastic process $\{\xi + X_t\eta_T; t \geq 0\}$ is also generated by $h(-\widetilde{\Delta_L})$.

In particular, if $\{X_t^\gamma; t \geq 0\}$, $0 < \gamma \leq 2$, is a strictly stable process with the characteristic function $c_{X_t^\gamma}(z)$ of X_t^γ given by $c_{X_t^\gamma}(z) = e^{-t|z|^\gamma}$, then $\{\xi + X_t^\gamma\eta_T; t \geq 0\}$ is generated by $-(-\widetilde{\Delta_L})^\gamma$.

5. A relationship to an infinite dimensional Ornstein-Uhlenbeck process

Put

$$[E]_{q,N} = \left\{ \varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E); \sum_{n=0}^{\infty} \alpha_N(n) e^{n^2/2} |f_n|_q^2 < \infty, \ \mathrm{supp}(f_n) \subset T, n = 0, 1, 2, \dots \right\}$$

for $q \geq 0$ and $N \geq 0$. Define a space $\overline{[E]_{q,N}}$ by the completion of $[E]_{q,N}$ with respect to the norm $|| \cdot ||_{\overline{[E]_{q,N}}}$ given by

$$||\varphi||_{\overline{[E]_{q,N}}} = \left(\sum_{n=0}^{\infty} \alpha_N(n) e^{n^2/2} |f_n|_q^2 \right)^{1/2}$$

for $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)^*$. Then $\overline{[E]_{q,N}}$ is a Hilbert space with norm $|| \cdot ||_{\overline{[E]_{q,N}}}$. It is easily checked that $\overline{[E]_{q,N}} \subset (E)_q$ for any $q \geq 0$. Put $\overline{[E]_{\infty,N}} = \bigcap_{q \geq 0} \overline{[E]_{q,N}}$ with the projective limit topology and also put $\overline{[E]_{\infty,\infty}} = \bigcap_{N \geq 1} \overline{[E]_{\infty,N}}$ with the projective limit topology.

Define an operator K on $\overline{[E]_{\infty,\infty}}$ by

$$K[\Phi] = S^{-1}[S[\Phi](e^{i\xi})].$$

Then we have the following:

Proposition 8. *Let $p \geq 1$. Then the operator K is a continuous linear operator from $\overline{[E]_{\infty,\infty}}$ into $\mathbf{E}_{-p,\infty}$.*

Proof. Let $p \geq 1$. Then for each $\ell \geq 1$ we can calculate the norm $|||K[\varphi]|||_{-p,N}^2$ of $K[\varphi]$ for $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in \overline{[E]_{\infty,\infty}}$ as follows:

$$\begin{aligned} |||K[\varphi]|||_{-p,N}^2 &= \sum_{n=0}^{\infty} \alpha_N(n) ||\langle : (e^{ix})^{\otimes n} :, f_n \rangle||_{-p}^2 \\ &\leq \sum_{n=0}^{\infty} \alpha_N(n) \sum_{\ell=0}^{\infty} \ell! \sum_{k_1,\dots,k_\ell=0}^{\infty} \prod_{j=1}^{\ell} (2k_j+2)^{-2p} \left| \sum_{|\nu|=\ell} \frac{1}{\nu!} \langle F_\nu, e_{k_1} \otimes \cdots \otimes e_{k_\ell} \rangle \right|^2, \end{aligned}$$

where $\nu = (\nu_1, \dots, \nu_n) \in \mathbf{N} \cup \{0\}$, $|\nu| = \nu_1 + \cdots + \nu_n$, $\nu! = \nu_1! \dots \nu_n!$ and $F_\nu = \int_{\mathbf{R}^n} f(\mathbf{u}) \hat{\otimes}_{j=1}^{n} \delta_{u_j}^{\hat{\otimes}\nu_j} d\mathbf{u}$. Since there exists $q \geq 0$ such that

$$\begin{aligned} &\sum_{k_1,\dots,k_\ell=0}^{\infty} \prod_{j=1}^{\ell} (2k_j+2)^{-2p} \left| \sum_{|\nu|=\ell} \frac{1}{\nu!} \langle F_\nu, e_{k_1} \otimes \cdots \otimes e_{k_\ell} \rangle \right|^2 \\ \leq \ & |f_n|_q^2 n^{2\ell} \left(\sum_{|\nu|=\ell} \frac{1}{\nu!} \right)^2 \left(\sum_{k=0}^{\infty} (2k+2)^{-2p} |e_k|_{-q}^2 \right)^{\ell}, \end{aligned}$$

we get that

$$\begin{aligned}|||K[\varphi]|||^2_{-p,N} &\leq \sum_{n=0}^{\infty} \alpha_N(n) e^{n^2 \sum_{k=0}^{\infty}(2k+2)^{-2(p+q)}} |f_n|_q^2 \\ &\leq \sum_{n=0}^{\infty} \alpha_N(n) e^{n^2/2} |f_n|_q^2.\end{aligned}$$

This is nothing but the inequality:

$$|||K[\varphi]|||_{-p,N} \leq ||\varphi||_{\overline{[E]_{q,N}}}.$$

Thus the proof is completed. □

Regarding K as an operator from $\overline{[E]_{\infty,\infty}}$ onto $K[\overline{[E]_{\infty,\infty}}]$, it is a bijection. Define a norm $[\![\cdot]\!]_{-p,q,N}$ on $K[\overline{[E]_{\infty,\infty}}]$ by

$$[\![\Phi]\!]_{-p,q,N} = \sqrt{||K^{-1}\Phi||^2_{q,N} + |||\Phi|||^2_{-p,N}}$$

for $\Phi \in K[\overline{[E]_{\infty,\infty}}]$. Let $\mathcal{K}_{-p,q,N}$ be the completion of $K[\overline{[E]_{\infty,\infty}}]$ with respect to $[\![\cdot]\!]_{-p,q,N}$. With the projective limit space $\mathbf{K}_{-p} = \bigcap_q \bigcap_N \mathcal{K}_{-p,q,N}$ and the inductive limit space $\mathbf{K}_{-\infty} = \bigcup_p \mathbf{K}_{-p}$, we have the following:

Propositon 9. *The operator K is a homeomorpism from $\overline{[E]_{\infty,\infty}}$ onto $\mathbf{K}_{-\infty}$.*

The operator K implies a relationship between $\overline{\Delta_L}$ and the number operator $\mathcal{N}$ on $(E)^*$ given by

$$\mathcal{N}\Phi = \sum_{n=0}^{\infty} n\mathbf{I}_n(f_n) \quad \text{for} \quad \Phi = \textstyle\sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)^*.$$

The operator K implies also a relationship between the semigroup $\{G[X_t^1]; t \geq 0\}$ and the E^*-valued Ornstein-Uhlenbeck process:

$$U_t^x = e^{-t}x + \sqrt{2}\int_0^t e^{-(t-s)} d\mathbf{B}(s), \quad t \geq 0,$$

where $\{\mathbf{B}(t); t \geq 0\}$ is a standard E^*-valued Wiener process starting at 0. Since $\overline{[E]_{\infty,\infty}}$ is in (E), we can apply the same proofs of Proposition 5 and Theorem 6 in [27] to get the following results.

Proposition 10. *For any $\varphi \in \overline{[E]_{\infty,\infty}}$ we have*

$$\overline{\Delta_L} K[\varphi] = -\frac{1}{|T|} K[\mathcal{N}[\varphi]].$$

Theorem 11. *For any* $\varphi \in \overline{[E]_{\infty,\infty}}$ *we have*

$$G[X_t^1]K[\varphi](x) = K[E[\varphi(U^x_{t/|T|})]].$$

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